91578 READINESS CHECK

Complete the problems on the next five pages as a 'warm up' before attempting the six practice achievement standards in our PEA 3 Differentiation Booklet.

QUESTION ONE

Differentiate the following.

(a) $y = 2\sqrt{x} - \frac{2}{x}$

(b) $y = 2(3x + 1)^3$

(c) $y = \sqrt{1 - x^2}$

(d) $y = {}^-4e^{2x-1}$

(e) $y = 5\ln(4x + 3)$

(f) $y = 2\cos 4x$

(g) $y = 6\sec\left(\frac{x}{3}\right)$

(h) $y = e^{\tan x}$

(i) $y = 3x^2 \ln x$

(j) $y = \frac{3x + 1}{2x - 3}$

QUESTION TWO

(a) Find the gradient of the tangent to the curve $y = \frac{2}{x^2 - 1}$ at the point (3, 0.25).

(b) Find the gradient of the normal to the curve $y = 4\sqrt{x} - x$ at $x = 9$.

QUESTION THREE

(a) Find the equation of the tangent to the curve $f(x) = (2x + 1)^4$ at the point $(^-1, 1)$.

(b) Find the equation of the normal to the curve $f(x) = 2 \sin \frac{1}{2}x$ at $x = \frac{2\pi}{3}$.

QUESTION FOUR

(a) Find the coordinates of the stationary points of the curve $y = x + \frac{1}{x}$.

(b) Find the coordinates of the stationary points of the curve $y = \frac{4x}{x^2 + 1}$.

QUESTION FIVE

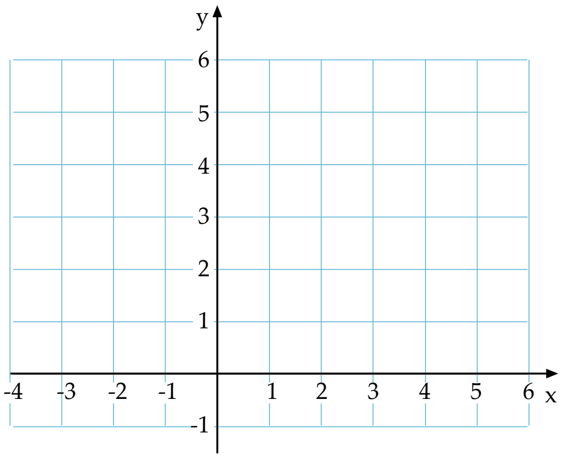
(a) Find the x coordinate(s) of the point of inflection of the graph $f(x) = 9\ln(x+3) + \frac{x^2}{8}$.

(b) When is the curve $f(x) = 2x^2 - \ln 2x$ increasing?

QUESTION SIX

(c) On the axis below, sketch a graph of f(x) that:

- is continuous for $^{-}3 < x < 0$ and $0 < x < 4$ and is discontinuous when $x = 0$.
- $f''(x) > 0$ for $^{-}3 < x < 0$.
- has $f'(x) = 0$ at $(^{-}1, 1)$.
- has $\lim\limits_{x \to 0} f(x) = 2$, but $f(0) = 3$.
- is not differentiable at (1, 4).
- has $f'(x) < 0$ for $x > 1$.



QUESTION SEVEN

(a) If $x = \dfrac{2}{t}$ and $y = 4t^2$, find the gradient of the normal at $t = 1$.

(b) If $x = 4\cos\theta$ and $y = 2\tan\theta$, find the gradient of the tangent at $\theta = \dfrac{\pi}{4}$.

(c) If $x = \sin^2\theta$ and $y = 2\cos\theta$, find the gradient of the normal at $\theta = \dfrac{\pi}{3}$.

QUESTION EIGHT

(a) The height of a kite as it soars into the air can be modelled by the function

$$H = \left(\frac{t}{6}+1\right)^3$$

where H is the height in metres and t is the time in seconds.

What is the rate of change in the height of the kite after 8 seconds?

(b) Coffee is poured into a cup and allowed to cool to room temperature. The temperature T of the liquid after t minutes is modelled by the formula

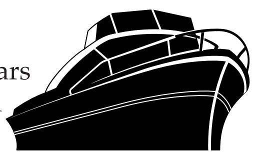
$$T = 70e^{-0.3t} + 25.$$

What is the rate of change of the temperature after 4 minutes?

QUESTION NINE

(a) The cost of running a boat on a trip is given by

$$C = 4.5v^2 + \frac{1200}{v}, \ v \neq 0$$



where C is the cost in dollars and v is the average speed in km/hr.

Find the average speed for which the cost is a minimum and find the minimum cost. (You may assume $\frac{d^2C}{dv^2} > 0$).

(b) A museum is holding a special exhibition. It records the number of visitors over several weeks. The number of people who attend can be modelled by the equation $P = 6000te^{-t/3}$, where t is the time in weeks.

In which week does the maximum number of visitors attend the exhibition and how many people attend?

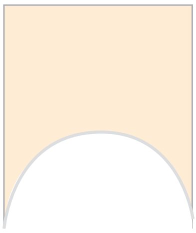
QUESTION TEN

A spherical balloon is inflated at the rate of 1.15 m^3 per minute.

What is the volume of the balloon when the radius is increasing at 0.15 m per minute?

QUESTION ELEVEN

Sophie wants to design an unusual noticeboard for her room. The design is a rectangle with a semicircle of radius r cut out from the bottom.



She wants the area of the noticeboard to be 2000 cm^2.

What is the minimum perimeter of the noticeboard?

QUESTION TWELVE

The sum of two numbers, x and y, is a fixed number, S.

The product of x and the square root y is to be as large as possible.

Use differentiation to show that the product is a maximum when $x = \frac{2S}{3}$.

Answers – Readiness Check

Question One:

(a) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}} + \dfrac{2}{x^2}$

(b) $\dfrac{dy}{dx} = 6(3x + 1)^2 \,.\, 3 = 18(3x + 1)^2$

(c) $\dfrac{dy}{dx} = 0.5(1 - x^2)^{-1/2} . {}^-2x = \dfrac{{}^-x}{\sqrt{1-x^2}}$

(d) $\dfrac{dy}{dx} = {}^-4e^{2x-1} \,.\, 2 \qquad = {}^-8e^{2x-1}$

(e) $\dfrac{dy}{dx} = \dfrac{5}{4x+3} \,.\, 4 \qquad = \dfrac{20}{4x+3}$

(f) $\dfrac{dy}{dx} = {}^-2 \sin 4x \,.\, 4 = {}^-8 \sin 4x$

(g) $\dfrac{dy}{dx} = 2\sec\left(\dfrac{x}{3}\right)\tan\left(\dfrac{x}{3}\right)$

(h) $\dfrac{dy}{dx} = e^{\tan x} \,.\, \sec^2 x$

(i) $\dfrac{dy}{dx} = 3x^2 \,.\, \dfrac{1}{x} + 6x \ln x = 3x + 6x \ln x$

(j) $\dfrac{dy}{dx} = \dfrac{3(2x-3) - 2(3x+1)}{(2x-3)^2} = \dfrac{{}^-11}{(2x-3)^2}$

Question Three:

(a) $f'(x) = 4(2x + 1)^3 \,.\, 2 = 8(2x + 1)^3$

At $x = {}^-1$, $f'(x) = {}^-8$

Equation: $y - 1 = {}^-8(x + 1)$

$y = {}^-8x - 7$

(b) $f'(x) = \cos 0.5x$ At $x = \dfrac{2\pi}{3}$, $f'(x) = 0.5$

so perpendicular gradient $= {}^-2$.

At $x = \dfrac{2\pi}{3}$, $f(x) = 1.732$ or $\sqrt{3}$.

Eqn: $y - 1.732 = {}^-2(x - \dfrac{2\pi}{3})$

$y = {}^-2x + 5.92$

Question Five:

(a) $f'(x) = \dfrac{9}{x+3} + \dfrac{x}{4}$

$f''(x) = \dfrac{{}^-9}{(x+3)^2} + \dfrac{1}{4}$

At a point of inflection, $f''(x) = 0$

$\dfrac{{}^-9}{(x+3)^2} + \dfrac{1}{4} = 0$

$\dfrac{{}^-9}{(x+3)^2} = \dfrac{{}^-1}{4}$

${}^-(x + 3)^2 = {}^-36$

$x = 3$ is the only solution.

(b) $f'(x) = 4x - \dfrac{1}{x}$

$4x - \dfrac{1}{x} = 0$

$4x^2 = 1$

$x^2 = \dfrac{1}{4}$

$x = \dfrac{1}{2}$ so increasing when $x > \dfrac{1}{2}$.

Question Two:

(a) $\dfrac{dy}{dx} = {}^-2(x^2 - 1)^{-2} \times 2x$

$= \dfrac{{}^-4x}{(x^2-1)^2}$

At $x = 3$, $\dfrac{dy}{dx} = \dfrac{{}^-3}{16}$

(b) $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{x}} - 1$

At $x = 9$, $\dfrac{dy}{dx} = \dfrac{{}^-1}{3}$

Normal has gradient = 3

Question Four:

(a) $\dfrac{dy}{dx} = 1 - \dfrac{1}{x^2}$

$1 - \dfrac{1}{x^2} = 0$

$x^2 = 1$

$x = \pm 1$

Max at $({}^-1, {}^-2)$
and min at $(1, 2)$

(b) Using quotient rule

$\dfrac{dy}{dx} = \dfrac{4(x^2+1) - 8x^2}{(x^2+1)^2}$

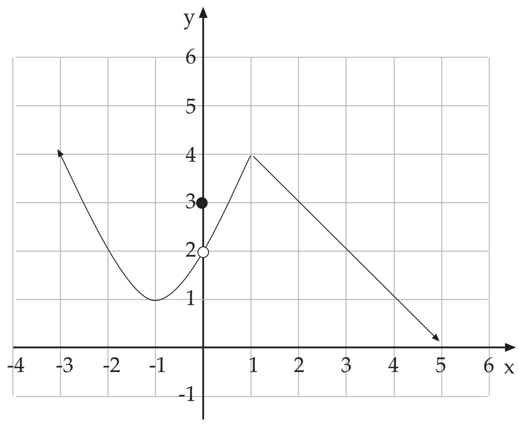
and $\dfrac{4(x^2+1) - 8x^2}{(x^2+1)^2} = 0$

${}^-4x^2 + 4 = 0$

$x = 1, {}^-1$

Coordinates are $(1, 2)$
and $({}^-1, {}^-2)$

Question Six:



Answers – Readiness Check

Question Seven: (a) $\frac{dx}{dt} = \frac{^-2}{t^2}, \; \frac{dy}{dt} = 8t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = {}^-4t^3$$

At $t = 1, \; \frac{dy}{dx} = {}^-4$

Normal has gradient $\frac{1}{4}$.

(b) $\frac{dx}{d\theta} = {}^-4\sin\theta \quad \frac{dy}{d\theta} = 2\sec^2\theta$

$$\frac{dy}{dx} = \frac{2\sec^2\theta}{^-4\sin\theta} = \frac{^-2}{\cos^2\theta} \div 4\sin\theta$$

At $\theta = \frac{\pi}{4}, \; \frac{dy}{dx} = {}^-\sqrt{2}$

(c) $\frac{dx}{d\theta} = 2\sin\theta\cos\theta$

$$\frac{dy}{d\theta} = {}^-2\sin\theta$$

$$\frac{dy}{dx} = \frac{^-2\sin\theta}{2\sin\theta\cos\theta} = \frac{^-1}{\cos\theta}$$

At $\theta = \frac{\pi}{3}, \frac{dy}{dx} = {}^-2$

Gradient of normal $= \frac{1}{2}$

Question Ten:

$$\frac{dV}{dt} = 1.15 \text{ m}^3/\text{min}$$

$V = \frac{4}{3}\pi r^3$ and $\frac{dV}{dr} = 4\pi r^2$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{4\pi r^2} \times 1.15 = \frac{1.15}{4\pi r^2}$$

$\frac{1.15}{4\pi r^2} = 0.15$ and $r = 0.781$ m

Volume $= \frac{4}{3} \times \pi \times 0.781^3$

$= 1.995 \text{ m}^3$

Question Twelve:

Product $P = x\sqrt{y}$

$x + y = S \Rightarrow x = S - y$

$P = (S - y)y^{1/2}$

$P = Sy^{1/2} - y^{3/2}$

$$\frac{dP}{dy} = \frac{1}{2}Sy^{-1/2} - \frac{3}{2}y^{1/2}$$

$$\frac{1}{2}S - \frac{3}{2}y = 0$$

$S = 3y \Rightarrow y = \frac{S}{3}$ and $x = \frac{2S}{3}$

Question Eight: (a) $\frac{dH}{dt} = 3(\frac{t}{6}+1)^2 . \frac{1}{6}$

$$= \frac{1}{2}(\frac{t}{6}+1)^2$$

At $t = 8, \; \frac{dH}{dt} = 2.72$ m/s

(b) $\frac{dT}{dt} = 70e^{-0.3t} . {}^-0.3 = {}^-21e^{-0.3t}$

At $t = 4, \; \frac{dT}{dt} = {}^-6.325°$

Cooling at 6.3° per minute.

Question Nine: (a) $\frac{dC}{dv} = 9v - \frac{1200}{v^2}$

$$9v - \frac{1200}{v^2} = 0$$

$$9v^3 = 1200$$

$$v = \sqrt[3]{\frac{1200}{9}}$$

$v = 5.1$ km/hr

Minimum cost = \$352.34

(b) $\frac{dP}{dt} = 6000e^{\frac{-t}{3}} - 2000te^{\frac{-t}{3}}$ (prod. rule)

$$e^{\frac{-t}{3}}(6000 - 2000t) = 0$$

$t = 3$

Maximum number in 3rd week.

People = 6621 or 6622

Question Eleven:

Area: $2000 = 2rh - \frac{\pi r^2}{2}$

so $h = \frac{1000}{r} + \frac{\pi r}{4}$

Perimeter: $P = 2r + 2h + \pi r$

$$P = 2r + 2\left(\frac{1000}{r} + \frac{\pi r}{4}\right) + \pi r$$

$$P = 2r + \frac{2000}{r} + \frac{3\pi r}{2}$$

$$\frac{dP}{dr} = 2 - \frac{2000}{r^2} + \frac{3\pi}{2}$$

Setting $\frac{dP}{dr} = 0$ and solving

$r = 17.26$ cm and $h = 71.5$ cm

Minimum perimeter = 231.7 cm